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Loop Edge Estimation in 4-Regular Hamiltonian Graphs

Yale Madden

Western Kentucky University

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LOOP EDGE ESTIMATION IN 4-REGULAR
HAMILTONIAN GRAPHS

A Thesis
Presented to
The Faculty of the Department of Mathematics
Western Kentucky University
Bowling Green, Kentucky

In Partial Fulfillment
Of the Requirements for the Degree
Master of Science

By
Yale Madden
May, 2007

LOOP EDGE ESTIMATION IN 4-REGULAR
HAMILTONIAN GRAPHS

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Director of Thesis

Claus Lint
David K. Neal
Uta Fiege

Richard A. Br May 14, 2007
Dean, Graduate Studies and Research Date

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LOOP EDGE ESTIMATION IN 4-REGULAR HAMILTONIAN GRAPHS

Name: Yale Madden

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Directed by Claus Ernst
Department of Mathematics

Western Kentucky University

Abstract In knot theory, a knot is defined as a closed, non-self-intersecting curve embedded in three-dimensional space that cannot be untangled to produce a simple planar loop. A mathematical knot is essentially a conventional knot tied with rope where the ends of the rope have been glued together. One way to sample large knots is based on choosing a 4-regular Hamiltonian planar graph. A method for generating rooted 4-regular Hamiltonian planar graphs with n vertices is discussed in this thesis. In the generation process of these graphs, some vertices are introduced that can be easily eliminated from the resulting knot diagram. The main result of this thesis is the estimation of the expected number of loop edges in a 4-regular Hamiltonian planar graphs of n vertices; in particular, it is shown that the expected number of loop edges $L(n)$ in such a graph has asymptotic order $\frac{n}{6}$.

CHAPTER 1

Defining and Counting the Number of Graphs

1.1. Introduction

The motivation for this thesis arises in the area of knot theory. In knot theory, a *knot* is defined as a closed, non-self-intersecting curve embedded in three-dimensional space that cannot be untangled to produce a simple planar loop [11]. A mathematical knot is essentially a conventional knot that we tie with rope where the ends of the rope have been glued together. A configuration that can be deformed to a planar circle is called the *unknot*. A *link* is made up of two or more knots, including the trivial or unknot, that are linked together.

Imagine a knot or link suspended in three-space over a flat piece of paper with a light source shining down from above the knot. The shadow that the knot projects onto the paper provides a two-dimensional diagram of the knot. If in the diagram there is an intersection of the shadows of more than two strands, we may “wiggle” the light source until no such intersection exists; i.e., all intersections are of two strands only. Within this diagram, the intersection of two strands is called a *crossing*. Figure 1 shows an example of the simplest knot, the trefoil, which has three crossings. Note that in the diagram, we have deleted a small piece of one of the two strands at each crossing, so that one can tell which strand passes “over” the other. The minimal number of crossings in all possible projections of a knot or link is the most common

method for describing the sizes of knots [11]. In mathematical knot tables, knots are ordered according to how many crossings one needs in a diagram. Since the diagram of a knot shadow is two-dimensional, all knot diagrams can be viewed as 4-regular planar graphs. Each vertex in the graph arises from a crossing in the knot diagram.

A natural question in knot theory is the following: How much rope must we have to tie a particular knot? In order for this to become independent of the thickness of the rope, we define the radius of the rope to be one unit and express the "rope length" in terms of this unit [5]. In order to study this question numerically, one can employ an algorithm to estimate the rope length for a given knot diagram [5, 7]. This raises the question of how to pick a large knot diagram at random. One way to do this is by first selecting a 4-regular planar graph at random. In this thesis we describe a method for generating a large random 4-regular planar graph that is also Hamiltonian. The additional requirement to be Hamiltonian is required by the algorithm to estimate rope length described in [5]. In the generation process of these graphs, some vertices are introduced that result in crossings which can be easily eliminated from the resulting knot diagram. The main topic of this thesis is to count the expected number of such crossings. For a more detailed description of the algorithm, see [1, 5, 7].

In the remainder of this chapter, we give definitions of several basic terms and a definition of a *rooted Hamiltonian graph*. We then discuss the count of rooted Hamiltonian 4-regular planar graphs and give bounds on this count.

In the second chapter we discuss the use of prefix vectors in generating rooted Hamiltonian 4-regular planar graphs. We then discuss situations in which *loop edges*

arise and obtain bounds on the number of loop edges in rooted Hamiltonian 4-regular planar graphs.

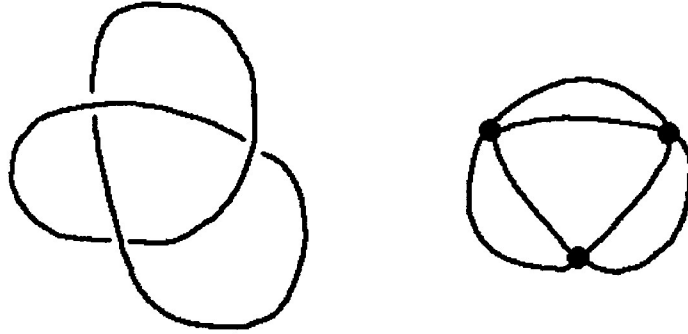


FIGURE 1. A knot diagram and a 4-regular planar graph that represent the three crossing trefoil, the simplest knot possible.

1.2. Definitions

This section lists several definitions that are essential to the topic of graph theory.

The definitions are taken from [13] but they can be found in any introductory text on Graph Theory.

A *graph* G is a set of vertices $V(G)$, a set of edges $E(G)$, and an assignment of a set of at most two vertices as endpoints of each edge. If an edge $e \in E$ has only one endpoint $v \in V$ associated with it, then both endpoints of e are v and e is called a *loop edge*. A *simple* graph is one with no loops or edges with the same endpoints. An *isomorphism* from a simple graph G to a simple graph H is a bijection, denoted $f : V(G) \rightarrow V(H)$, such that the edge from vertex u to v , denoted uv , is contained in $E(G)$ if and only if the edge $f(u)f(v)$ is contained in $E(H)$.

The term *incident* refers to a vertex v and edge e with v an endpoint of e . The *degree* of a vertex is the number of edges incident to v , where a loop edge is counted twice (once for each end).

A k -regular graph G has all vertex degrees equal to k .

A *path* is a simple graph whose vertices can be listed so that adjacent vertices are listed consecutively in the list; i.e., all vertices have degree two, with the exception of the two endpoints of the path, which have degree one.

A graph G is *connected* if for any two vertices u and v in G , there is a path in G with endpoints u and v .

A *cycle* or a *cyclic graph* is a 2-regular connected graph. We denote a cycle with n vertices by $C(n)$.

A *Hamiltonian cycle* is a cycle in a graph containing each vertex (of the graph). A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*.

A *drawing* of a graph G on a surface S (or an *embedding* of G on S) must satisfy the following conditions [8] :

- (i) The vertices of G are disjoint points on S .
- (ii) The drawing of an edge is a continuous path on S such that:
 - a) The drawing of an edge does not intersect itself except possibly at the endpoints of the edge.
 - b) The drawing of an edge does not intersect any vertices except at its endpoints.
 - c) The drawings of two edges do not intersect except possibly at their endpoints.

A *diagram* is a drawing of a graph G on a surface S [8].

A *planar graph* is a graph that has a drawing on the plane.

Finally, from any standard text in topology [9]:

A *homeomorphism* between two sets A and B , denoted $f : A \rightarrow B$, is a one-to-one, onto, continuous function f with a continuous inverse $f^{-1} : B \rightarrow A$.

1.3. Rooted Hamiltonian graphs

Definitions 1.3.1, 1.3.2 and 1.3.3 are taken from [2].

DEFINITION 1.3.1. A *rooted Hamiltonian graph* G consists of a pair (G, H) satisfying the following conditions:

- (i) G is a 4-regular planar graph in S^2 with a given Hamiltonian cycle H ;
- (ii) H contains a rooted edge; that is, one edge of H has an orientation. This edge defines an orientation on H that can distinguish the two disks bounded by H on S^2 as a disk on the right hand side (the bounded or inside disk) and a disk on the left hand side (the unbounded or outside disk) of H .

DEFINITION 1.3.2. Two rooted Hamiltonian graphs (G, H) and (G', H') are *equivalent* if there exists a function $f : (S^2) \rightarrow (S^2)$ such that:

- (i) f is an orientation preserving homeomorphism;
- (ii) $f(G) = G'$, $f(H) = H'$ and $f|_G$ is an isomorphism between the rooted graphs G and G' . That is, f is an isomorphism that maps the rooted edge of G to the rooted edge of G' while preserving its direction.

DEFINITION 1.3.3. A *standard diagram* of a rooted Hamiltonian 4-regular planar graph G is a drawing of G such that H is drawn as a circle with its vertices numbered

in a clockwise order around H and its first vertex v_1 (at the 12:00 o'clock position) of the circle. The rooted edge is the edge from v_1 to v_2 along the Hamiltonian cycle.

For an example of a standard diagram, see Figure 2.

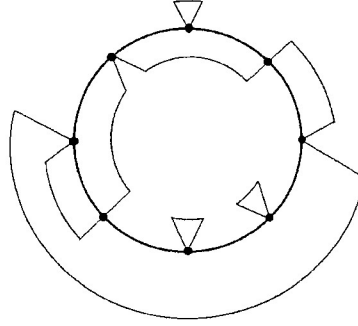


FIGURE 2. An example of a standard diagram.

The term *inside* will be used to denote the side of the cycle H that is bounded by H ; i.e., the inside of the circle H .

The term *outside* will be used to denote the side of the cycle H that is not bounded by H ; i.e., the outside of the circle H .

If we consider a graph G as a rooted Hamiltonian 4-regular planar graph with Hamiltonian cycle H , then we can classify the vertices of G into three categories. For each vertex v of G , v is incident to four edges. Two of these edges are contained in the Hamiltonian cycle H and two of the edges are not in H . The edges that are not in H lead us to the following three classifications:

(a) v is a *double-outside vertex*, or DOV. If v is a DOV, then v is incident to two edges, e_1 and e_2 , or one loop edge, l , where e_1 and e_2 or l lie entirely outside of H .

(b) v is a *double-inside vertex*, or DIV. If v is a DIV, then v is incident to two edges, e_1 and e_2 , or one loop edge, l , where e_1 and e_2 or l lie entirely inside of H .

(c) v is a *transition vertex*, or TV. If v is a TV, then v is incident to two edges, e_1 and e_2 , where, without loss of generality, e_1 lies inside of H and e_2 lies outside of H .

Similarly, for an edge e of G that is not on H , e is either an *inside edge* (IE) or an *outside edge* (OE), depending on whether it is inside or outside of H .

In the construction process of rooted Hamiltonian 4-regular graphs we need to use the following concept.

A *positive prefix vector* (PPV) is a binary string of 1s and 0s such that there are an equal number of 1s and 0s and the number of 0s is never greater than the number of 1s when considering an initial substring of 1s and 0s. The binary string 1011010010 is an example of a string with the positive prefix property. However, the string 1001010110 does not satisfy the positive prefix property because the number of 0s is greater than the number of 1s in the initial substring 100.

1.4. The number of rooted Hamiltonian 4-regular planar graphs

In this section we discuss the construction of rooted Hamiltonian 4-regular planar graphs and cite theorems that give the exact number of non-equivalent rooted Hamiltonian 4-regular planar graphs. The work described in this section is a summary of two prior Master's theses by D. High and O. Ascigil [1, 8]

The construction of a rooted Hamiltonian 4-regular planar graph G begins with a 2-regular Hamiltonian cycle. Let this cycle be denoted H . We designate the vertex types (DOV, DIV and TV) by attaching line segments to the respective vertices.

These line segments are called *edge endpoints* and will later be paired to form the edges

of G that are not in H . The DOVs will have attached to them two line segments on the outside of H . Likewise, the DIVs will have two edges attached on the inside of H . The TVs will have one edge attached on the inside of H and one edge on the outside of G . The edge endpoints on the outside of H will be referred to as *outside edge endpoints* or OEEPs and the edge endpoints on the inside of H will be called *inside edge endpoints* or IEEPs. For an example of this see Figure 3.

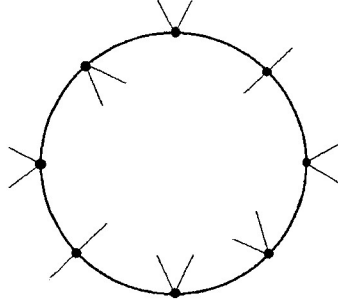


FIGURE 3. An illustration of the edge end points from the graph of Figure 2.

To demonstrate how the edge endpoints are connected, we consider the standard diagram from Figure 2. It will be convenient if we only consider one side of the cycle H . Without loss of generality, we will consider the inside. As we travel clockwise around H , there is a natural ordering of the vertices that comes from the pairing of IEEPs. As we travel we will build a binary string as follows: we assign the digit 1 to our string when we encounter the first edge endpoint in a connected pair of edge endpoints, we assign the digit 0 to our string when we encounter the second edge endpoint in a connected pair of edge endpoints. Since the IEEPs are paired, this process yields a binary string with an equal number of 1s and 0s. Due to the fact that the edges are paired and they do not intersect each other, this binary string is a PPV

[2]. The PPV for the inside edges in Figure 2 is 11010100. Similarly, a PPV can be assigned to the outside edges.

In the same manner that we can build an inside and outside PPV from the standard diagram of a rooted Hamiltonian 4-regular planar graph, we can construct a rooted Hamiltonian 4-regular planar graph from two PPVs of appropriate length given a particular arrangement of DOVs, DIVs and TVs. For an example, see Figure 4. We may conclude that for any standard diagram of a rooted 4-regular Hamiltonian planar graph with a given arrangement of vertex types, there exists one and only one PPV pair that corresponds to that diagram. These observations lead to the following algorithm for generating rooted Hamiltonian 4-regular planar graphs with n vertices [2].

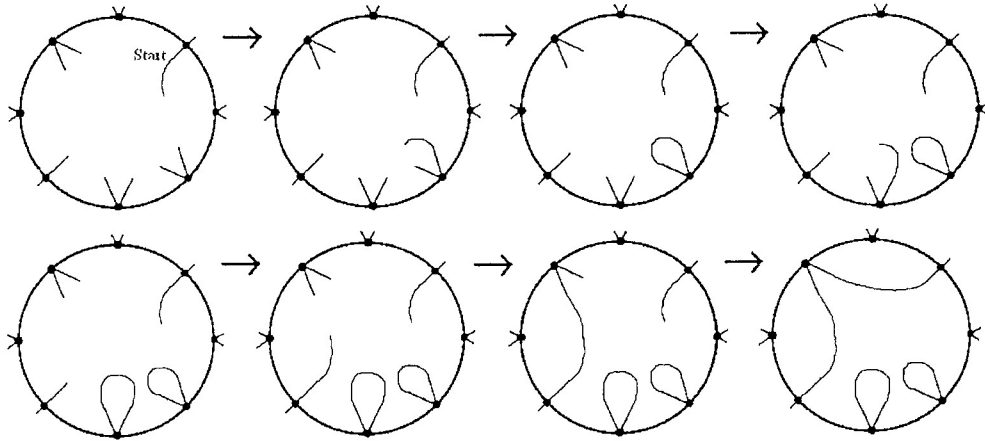


FIGURE 4. Given the PPV 11010100 for the IEEPs, the inside edges from the graph of 2 are reconstructed.

The algorithm has two steps. The first step is to choose the number of vertices of each of the three vertex types (DIV, DOV and TV) and randomly arrange the vertices

along a Hamiltonian cycle H_n of n vertices. The second step is to connect the inside edges and outside edges separately using two randomly chosen PPVs.

Step 1 - Vertex Arrangement: First, we decide how many vertices of each type (DOV, DIV and TV) we will place and where to place them on our 2-regular Hamiltonian cycle of n vertices, denoted H_n .

In choosing the vertex types, we first note that there cannot be an odd number of TVs. If we only consider the inside edges of a completed standard diagram, there must be an even number of inside edge endpoints for the edges to be connected correctly. Since there are an even number of edge endpoints coming from the DIVs, an odd number of TVs would create an odd number of edge endpoints. To ensure that the number of TVs is even, we choose $2t$ TVs from the n vertices. The number of ways to choose $2t$ TVs from n vertices for a given t is $\binom{n}{2t}$.

We now have $n - 2t$ vertices remaining from which to choose the DIVs and DOVs. If we choose p vertices to be DIVs, the remaining $n - 2t - p$ vertices must all become DOVs. Similarly, if we choose p vertices to be DOVs, the remaining $n - 2t - p$ vertices must all become DIVs. With either choice, the number of ways to choose p double vertices is $\binom{n-2t}{p} \binom{n-2t-p}{n-2t-p} = \binom{n-2t}{p}$. We will opt to choose p DIVs. The total number of ways to arrange $2t$ TVs and p DIVs is:

$$\binom{n}{2t} \binom{n-2t}{p}$$

Figure 2 is an example of a standard diagram with $n = 8$, $t = 1$ and $p = 3$.

At this stage in the construction of a 4-regular Hamiltonian planar graph, we know the number of vertices of each type and where along H they will be placed. It is

not yet known which vertices will be connected; we only know which vertices' endpoints will be incident to inside edges and to outside edges.

Step 2 - Edge Assignment: We now connect the inside edges. Recall that there are $2t$ TVs and p DIVs. Since each DIV produces two IEEPs, we can see that there are $2t + 2p$ IEEPs. Since there are $2t + 2p$ IEEPs, we will need a PPV of length $2t + 2p$ to connect the IEEPs; this PPV will be called the *inside vector*. Since there are the same number of digits in the PPV as there are IEEPs, each digit in the PPV corresponds to one IEEP. Before we begin connecting the IEEPs, we must first determine the starting edge endpoint. This determination is made by beginning at v_1 and traveling clockwise until the first IEEP is encountered; this will be marked as the starting IEEP. From the starting IEEP, we now apply the PPV. If a 1 digit is encountered in the PPV, we start an edge from the current IEEP, leaving it open to be closed later. When a 0 digit is encountered, we close an edge at the current IEEP with the most recently opened, but not yet closed, edge. We continue in this manner, beginning at the starting IEEP with the first digit in the PPV and making the corresponding assignment of edges as we traverse around the Hamiltonian cycle H_n and along the PPV concurrently. This process is similar to the pairing of parentheses in a mathematical expression. For an example, see Figure 4.

It is known that the number of PPVs of length $2k$ is $\frac{1}{k+1} \binom{2k}{k}$, which is also commonly called a Catalan number [2, 8, 10]. (See Chapter 2 for an explanation of this formula.) Since the length of the inside vector is $2t + 2p$, we may conclude that there are $\frac{1}{t+p+1} \binom{2t+2p}{t+p}$ ways to choose the inside vector.

We go about connecting the outside edges in the same manner. Given that the total number of edge endpoints for a rooted Hamiltonian 4-regular planar graph of n vertices is $2n$, we must use an *outside vector* of length $2n - 2t - 2p$. Similar to the inside vector, there are $\frac{1}{n-t-p+1} \binom{2n-2t-2p}{n-t-p}$ ways to choose the outside vector. It follows then immediately that there are

$$\frac{1}{(t+p+1)} \binom{2t+2p}{t+p} \frac{1}{(n-t-p+1)} \binom{2n-2t-2p}{n-t-p}$$

ways to connect all of the edges [8].

Now that we have connected all of the edges, our original 2-regular Hamiltonian cycle has become a 4-regular planar Hamiltonian graph. We have shown that given values for t and p , there are $\binom{n}{2t} \binom{n-2t}{p}$ ways to arrange the points and

$\frac{1}{(t+p+1)} \binom{2t+2p}{t+p} \frac{1}{(n-t-p+1)} \binom{2n-2t-2p}{n-t-p}$ ways to connect all the edges.

From Step 1 and Step 2, we arrive at the conclusion that for a fixed t and p , there are $\binom{n}{2t} \binom{n-2t}{p} \frac{1}{t+p+1} \binom{2t+2p}{t+p} \frac{1}{n-t-p+1} \binom{2n-2t-2p}{n-t-p}$ total graphs. As discussed in [8], t must be chosen so that $2t \in \{0, 2, 4, \dots, n\}$ if n is even, and $2t \in \{0, 2, 4, \dots, n-1\}$ if n is odd. Hence $t \in \{0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ will ensure that $2t$ is even and in the proper interval.

Next, we must choose p DIVs from zero up to a maximum of $n - 2t$. Recall that once t and p are chosen, the remaining vertices all become DOVs, hence the number of DOVs will depend on the values for t and p . Exhausting all possibilities of t and p , we obtain the following theorem. For a proof and more details on the construction see [8].

THEOREM 1.4.1. *The number $X(n)$ of non-equivalent rooted Hamiltonian 4-regular planar graphs of n vertices is*

$$X(n) = \sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} \binom{n}{2t} \binom{n-2t}{p} \frac{\binom{2t+2p}{t+p}}{(t+p+1)} \frac{\binom{2n-2t-2p}{n-t-p}}{(n-t-p+1)}$$

1.5. Bounds on the number of graphs

In this section, we cite estimates on the upper and lower bounds of the number of non-equivalent rooted Hamiltonian 4-regular planar graphs of n vertices. Let

$$f(n, t, p) = \binom{n}{2t} \binom{n-2t}{p} \frac{\binom{2t+2p}{t+p}}{(t+p+1)} \frac{\binom{2n-2t-2p}{n-t-p}}{(n-t-p+1)}$$

so that

$$X(n) = \sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p).$$

Numerical evidence collected in [8] suggests that

$$X(n) \approx \frac{4}{\pi} \frac{12^n}{n^3}.$$

The following recent theorem [2] gives the asymptotic order of $X(n)$:

THEOREM 1.5.1. *The number $X(n)$ of non-equivalent rooted Hamiltonian 4-regular planar graphs of n vertices has asymptotic order*

$$X(n) \sim \frac{4}{\pi} \frac{12^n}{n^3}.$$

That is, $\lim_{n \rightarrow \infty} \frac{4}{\pi} \frac{12^n}{n^3 X(n)} = 1$.

It is also shown in [8] that the absolute maximum of $f(n, t, p)$ occurs at $t = n/6$ and $p = n/3$ when $(n \bmod 6) = 0$. Moreover, in [2] it is shown that:

$$f(n, t, p) \geq \frac{12\sqrt{3}}{\pi^2} \frac{12^n}{n^4} * \frac{1}{e^{(9x^2+3y^2)/n}},$$

where $t = \frac{n}{6} + x$, $p = \frac{n}{3} + y$, and $0 \leq |x|, |y| \leq n^{\frac{3}{5}}$.

CHAPTER 2

Counting Loop Edges

2.1. Introduction

This section contains new results that constitute my thesis work. The goal is to obtain an estimate on the expected number of loop edges in a randomly chosen Hamiltonian 4-regular planar graph of n vertices. When the graph is turned into a knot diagram, the loop edges are redundant because a simple deformation of the knot eliminates the loop edge together with the crossing, see Figure 5. In knot theory, such a deformation is called a Reidemeister I move [3]. Thus, counting loop edges is an estimate of redundant complexity if one considers these graphs as knot diagrams. Certainly one can generate random graphs without loop edges, but such algorithms do not have many of the nice properties as the algorithm discussed in Chapter 2. For a discussion of this technique, see [6, 7, 2].

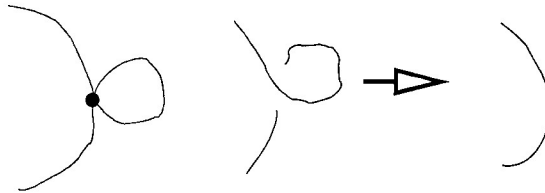


FIGURE 5. There is a loop edge at the vertex shown in the graph to the far left. The knot or link diagram corresponding to this graph has a crossing that can be removed by a simple twist.

Our goal is to prove the following result:

THEOREM 2.1.1. *The expected number of loop edges $L(n)$ in a randomly chosen rooted Hamiltonian 4-regular planar graph of n vertices has asymptotic order*

$$L(n) \sim \frac{n}{6}.$$

In order for a loop edge to occur, two things must happen. There must be an occurrence of a 10 (read one zero) pair in the PPV corresponding to a particular side. The 10 pair must also occur in the proper place within the PPV, with the 1 corresponding to the first edge endpoint of a DIV/DOV and the 0 to the second edge endpoint. In order to count the expected number of loop edges, we need to understand the counting of prefix vectors in general and then adapt this count to our special case; i.e., the count of 10 pairs in prefix vectors.

In Section (2.2) we describe this counting of prefix vectors; the method described is based on an algorithm described in [10]. In Section (2.3) we adapt this count to establish the number of prefix vectors with k 10 pairs. The result of Section (2.3) is used in Sections (2.5) and (2.4) to establish lower and upper bounds on the expected number of loop edges.

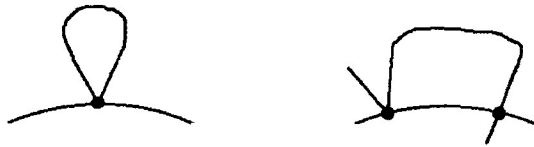


FIGURE 6. The figure on the left is an example of a placement of a 10 pair leading to a loop edge. The figure on the right is an example of a placement *not* leading to a loop edge.

2.2. Counting Prefix Vectors

In this section we discuss the process of transforming a balanced binary string, which is one having an equal number of 1s and 0s, into a PPV. This process is done with the help of a *bridge* graph and follows the argument laid out in [1, 10].

We construct a *bridge* from a binary string of length $2n$ as follows. We begin with $2n + 1$ evenly spaced tick marks on a dashed line, these will serve as our horizontal step size and will be used to horizontally space $2n + 1$ vertices, v_1 through v_{2n+1} . Starting at v_1 , on the dashed line, and reading left to right, whenever a 1 is encountered in the binary string, we add an up-step to the bridge. Whenever a 0 is read, we add a down-step to the bridge. In Figure 7 we see the bridge representation of the binary string 10010110. Since there are an equal number of 1s and 0s in the binary string, vertices v_1 and v_{2n+1} are both on the dashed line.

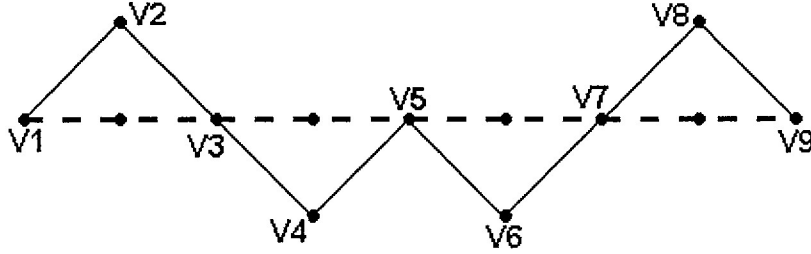


FIGURE 7. A bridge representation of the binary string 10010110.

To obtain a balanced binary string, we may place n 1s into $2n$ open slots. There are n remaining slots that will be filled with zeros, giving us a balanced binary string. There are $\binom{2n}{n}$ ways to place n 1s into $2n$ slots and $\binom{2n-n}{n} = \binom{n}{n} = 1$ way to place n 0s into n slots, so there are $\binom{2n}{n}$ balanced binary strings of length $2n$. Equivalently, there are $\binom{2n}{n}$ different bridge presentations.

After we have a binary string, and thus a bridge, of appropriate length, we then add a down-step to the end of the bridge, effectively adding a 0 to the binary string. The addition of this down-step is important to establishing a count for prefix vectors as will be explained later. In Figure 8 we can see the resulting bridge from the addition of a down-step to the end of the bridge in Figure 7. The number of bridges of length $2n + 1$ and whose last step is a down step is $\binom{2n}{n}$ as well [1, 10].

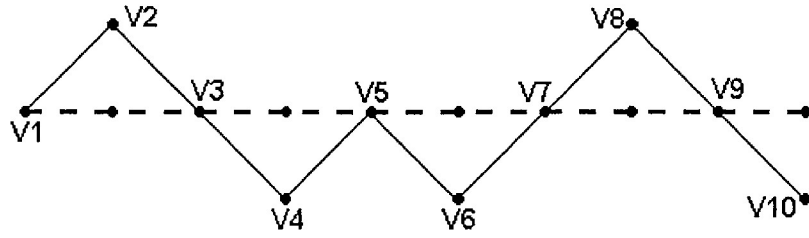


FIGURE 8. The bridge after the last down step is added to the bridge in Figure 7.

It is easy to see visually from the bridge representation if the corresponding binary string has the positive prefix property. If the bridge contains no vertices at a lower height than v_1 , except for the last vertex in the bridge, then the equivalent binary string has the positive prefix property. If the binary string is found to not possess the positive prefix property, then we follow the transformation process explained below.

The transformation of a bridge into a bridge whose binary string possesses the positive prefix property is done by forming *conjugate paths* of the original bridge.

DEFINITION 2.2.1. *A conjugate path of a bridge can be formed by applying the following transforms:*

- a) Locate a down-step in the bridge, denote the vertex number after the down-step as v_x .
- b) Delete the path P starting from v_1 and ending at v_x .
- c) Append P to the end of the bridge.

In Figure 9 we can see the conjugate path resulting from the deletion of the path from v_1 to v_6 in Figure 8 and appending it to the end of the bridge.

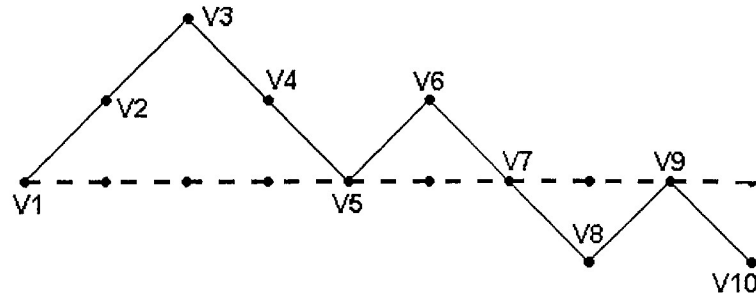


FIGURE 9. A conjugate path resulting from the deletion of the path from v_1 to v_6 in Figure 8 and appending it to the end of the bridge. The vertices have been renumbered as well.

For each down-step in a bridge, there exists a conjugate path of the bridge using the vertex after the down-step. Since a bridge has $n + 1$ down-steps, there are $n + 1$ conjugate paths for any bridge.

Exactly one of these $n + 1$ conjugate paths of a bridge corresponds to a binary string with the positive prefix property, as shown in [4]. In order to form a binary string with the positive prefix property, it is first necessary to form a conjugate path by using a path from v_1 to a local minimum. If we do not use a local minima, the resulting conjugate path will begin with a down-step, which immediately violates the positive prefix property. Moreover, we must use a path from v_1 to the first absolute minimum encountered in the bridge when reading from left to right [1, 10]. From the

bridge in Figure 8, we obtain a binary string with the positive prefix property by forming a conjugate path using the path from v_1 to v_4 and applying the transform process in Definition (2.2.1); the bridge representation of the string with the positive prefix property is shown in Figure 10. The last down step, from v_9 to v_{10} , and the corresponding 0 in the binary string are now deleted. There is only one left most absolute minimum in a bridge, thus there is only one conjugate path leading to a binary string with the positive prefix property.

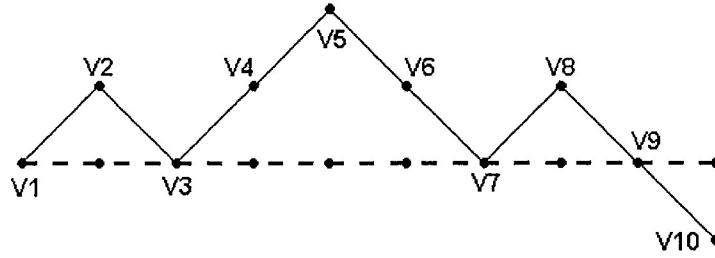


FIGURE 10. The conjugate path of the bridge in Figure 8 with the positive prefix property formed by deleting the path from v_1 to v_4 and appending it to the end of the bridge and renumbering the vertices.

Note that we can only use the first absolute minimum encountered in a bridge when reading from left to right. This is because of the addition of the down-step to the end of the original bridge that guarantees that there is a unique conjugate path of a bridge corresponding to a binary string with the positive prefix property. An example of this is taken from [1], the bridge in Figure 11 is the bridge representation of the binary string 100110010, note that the down-step has already been added to the end of the bridge. This bridge has absolute minima at vertices v_4 , v_8 and v_{10} .

Only selecting v_4 to form a conjugate path from the bridge in Figure 11 will yield a bridge with the positive prefix property. If we use v_8 , then the binary string

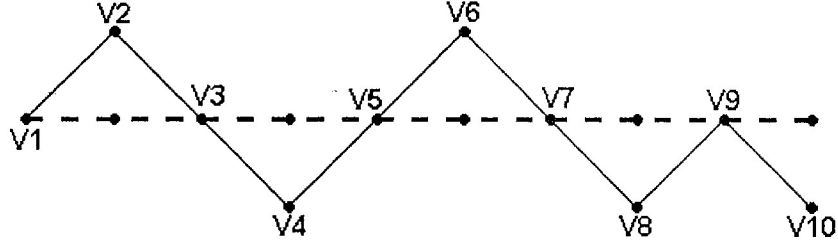


FIGURE 11. This bridge has absolute minima at vertices v_4 , v_8 and v_{10} . Only using v_4 for the conjugate path will lead to the positive prefix property.

represented by the resulting conjugate path will be 101001100, which is not a PPV.

Using v_4 instead, results in the binary string 110010100, which is a PPV [1].

Thus we have shown for a bridge with n up steps and $n + 1$ down steps, exactly one of these $n + 1$ conjugate paths forms a bridge whose corresponding binary string has the positive prefix property. Therefore, there are

$$\frac{1}{n+1} \binom{2n}{n}$$

prefix vectors of length $2n$, for further details see [1, 10].

2.3. Counting of Prefix Vectors with k 10 Pairs

A slight modification of the construction in Section (2.2) will yield the expected number of 10 pairs within a PPV. Recall that all PPVs are of even length, $2n$; i.e., each vector contains n 1s and n 0s. To obtain a positive prefix vector with k 10 pairs, we begin with the n 1s and divide them into k non-empty blocks. This is done by placing $k - 1$ dividers between the n 1s. There are $n - 1$ slots for the dividers, so there are $\binom{n-1}{k-1}$ ways to place the dividers.

As before, in our construction of a PPV we initially will use $n + 1$ 0s (for the same reasons as explained in Section (2.2)). Therefore, similar to the 1s, there are $\binom{n}{k-1}$ ways to divide the $n + 1$ 0s into k blocks.

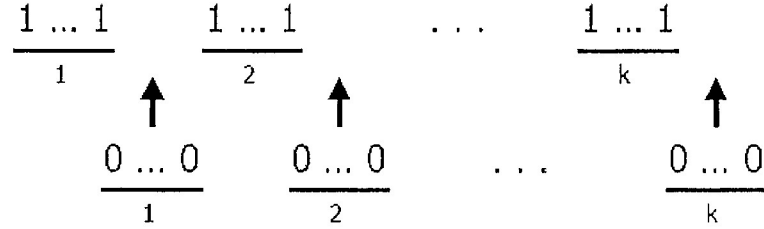


FIGURE 12. The blocks of 1s and 0s will be alternately placed as shown.

The blocks of 1s and 0s will be alternately placed as shown in Figure 12 to create a single binary string with length $2n + 1$ containing n 1s and $n + 1$ 0s and with k 10 pairs. This string most likely will not have the positive prefix property. To obtain the positive prefix property we can cyclically permute this string as explained in Section (2.2).

For the strings generated as in Figure 12, we know that they start with a block of 1s (or a single 1), and they also end with a 0 (or a block of 0s). Any local minimum in a bridge derived from one of these strings will occur after a block of 0s, including at the end of the bridge. We know that in order to generate a bridge with the positive prefix property (or equivalently a PPV), we must form the conjugate path by selecting the first vertex at the lowest point when traversing from left to right. We also know that there are k blocks of 0s in a binary strings with k 10 pairs, and thus k local minima in such a bridge. So there are k choices to form a conjugate path that may have the positive prefix property. Note that if a conjugate path is constructed by

choosing a 0 that is not at the end of a block, one obtains a vector that has either $k + 1$ blocks of 0s or no longer has k 10 pairs. Therefore only one choice for a conjugate path will have the positive prefix property; thus, there is only one of the k arrangements that will yield a PPV.

Now the last digit of the bridge can be dropped since it is not part of the PPV. In summary, only one of the k arrangements yields a PPV, and there are $\binom{n-1}{k-1} \binom{n}{k-1}$ ways to divide the n 1s and $n + 1$ 0s into k blocks. We have shown the following:

LEMMA 2.3.1. *There are $\binom{n-1}{k-1} \binom{n}{k-1} \frac{1}{k}$ PPVs of length $2n$ with k instances of a 10 pair.*

A PPV of length $2n$ must have at least one 10 pair (e.g. 11...1100...00), also it can have at most n 10 pairs (e.g. 1010...1010). Summing from 1 to n we have the number of PPVs of length $2n$ given by:

$$a(n) = \sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1} \frac{1}{k}$$

Recall from Chapter 1 that the number of PPVs of length $2n$ is also already known to be $\frac{1}{n+1} \binom{2n}{n}$. Therefore, we have shown the following lemma:

LEMMA 2.3.2. *The number $a(n)$ of positive prefix vectors of length $2n$ is*

$$a(n) = \sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1} \frac{1}{k} = \frac{1}{n+1} \binom{2n}{n}$$

Since each PPV has k 10 pairs for some k , if we multiply each term in the above sum by the number of 10 pairs for the k^{th} term, we obtain the total number of 10 pairs in all PPVs of length $2n$. That number is given by

$$b(n) = \sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1}$$

To simplify this sum, we consider the individual terms of the sum: $\binom{n-1}{k-1}\binom{n}{k-1}$.

Each term may be thought of as the number of ways to divide n 1s into k blocks and $n + 1$ 0s into k blocks, as in Figure 12. Thus, our sum is equivalent to the sum of the number of binary strings of length $2n$ with the same number of 0s and 1s, that begin with a one and contain k 10 pairs.

PROPOSITION 2.3.3. *The number, $b(n)$, of 10 pairs in all PPVs of length $2n$ is:*

$$b(n) = \sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1} = \binom{2n}{n} \frac{1}{2}$$

PROOF. We need to show that for any binary string v from the count $\binom{2n}{n} \frac{1}{2}$, v is also obtained from the count $\sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1}$, and vice versa.

The quantity $\binom{2n}{n} \frac{1}{2}$ is equivalent to the number of binary strings of length $2n$ that contain the same number of 1s and 0s and that begin with a 1. Any such string v also contains j blocks of 1s, for some j . Thus, there exists a string v' from the count $\sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1}$ such that $v = v'$.

Similarly, for any string v obtained from the count $\sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1}$, there exists a string v' obtained from the count $\binom{2n}{n} \frac{1}{2}$ such that $v = v'$. Thus, we have that

$$b(n) = \sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1} = \binom{2n}{n} \frac{1}{2}. \quad \square$$

PROPOSITION 2.3.4. *The expected number $c(n)$ of 10 pairs in a PPV of length $2n$ is $\frac{n+1}{2}$*

PROOF. The average number of 10 pairs in a PPV of length $2n$ is given by

$b(n)/a(n)$; hence,

$$\begin{aligned} c(n) = \frac{b(n)}{a(n)} &= \frac{\sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1}}{\sum_{k=1}^n \binom{n-1}{k-1} \binom{n}{k-1} \frac{1}{k}} \\ &= \frac{\binom{2n}{n} \frac{1}{2}}{\frac{1}{n+1} \binom{2n}{n}} \\ &= \frac{n+1}{2} \end{aligned}$$

□

As the next step, we estimate the probability that a 10 pair leads to a loop edge.

If we let p be the number of DIVs, then there are $2p$ edge endpoints belonging to the p DIVs. Also, let $2t$ be the number of edge endpoints from TVs. Then we have $2p + 2t = 2m_p$ total edge endpoints. This is the length of the PPV for the inside edge placement. To obtain a loop edge, the 1 in a 10 pair must occur at the first edge endpoint from a DIV.

Consider two separate PPVs, v_1 and v_2 , of length $2q$. Assume that v_1 has a 10 pair with the 1 as the i^{th} digit in the vector; also assume that v_2 has a 10 pair with the 1 as the j^{th} digit in the vector. Removing a 10 pair from a PPV does not affect the positive prefix property. Similarly, inserting a 10 pair into a PPV also preserves the positive prefix property. If we remove the 10 pair from each of the vectors v_1 and v_2 , we obtain two vectors of length $2q - 2$. There are a set number of PPVs of length $2q - 2$. Consider a PPV v_3 of length $2q - 2$. We may insert a 10 pair into v_3 at any position we choose and we will obtain a PPV of length $2q$. So there are the same number of PPVs of length $2q$ containing a 10 pair in the i^{th} position as there are PPVs of length $2q$ containing a 10 pair in the j^{th} position.

Thus, there are p equally likely placements of a 10 pair leading to a loop edge out of $2p + 2t - 1$ possible placements (the one in a 10 pair must be in a position ranging from 1 to $2p + 2t - 1$). So the probability of having a 10 pair in a placement leading to an inside loop edge is $\frac{p}{2p+2t-1}$. This quantity along with the expected number of 10 pairs in a PPV of length $2m_p$, $\frac{m_p+1}{2}$, gives us the expected number of inside loop edges in a graph with p DIVs and $2t$ TVs as

$$\frac{m_p + 1}{2} * \frac{p}{2p + 2t - 1} = \frac{t + p + 1}{2} * \frac{p}{2p + 2t - 1}$$

For the outside vector, let o be the number of DOVs. Then the length of the vector is $2m_o = 2o + 2t$. Similar to the argument above, the expected number of outside loop edges is

$$\frac{o + t + 1}{2} * \frac{o}{2o + 2t - 1}$$

So for a graph with n vertices and fixed t and p , we know that $n = o + p + 2t$. So we have the expected number of loop edges for such a graph as

$$\begin{aligned} g(n, t, p) &= \frac{t + p + 1}{2} * \frac{p}{2p + 2t - 1} + \frac{o + t + 1}{2} * \frac{o}{2o + 2t - 1} \\ &= \frac{t + p + 1}{4} * \frac{p}{p + t - \frac{1}{2}} + \frac{n - p - t + 1}{4} * \frac{n - p - 2t}{n - p - t - \frac{1}{2}} \\ &= \frac{t + p + 1}{t + p - \frac{1}{2}} * \frac{p}{4} + \frac{n - p - t + 1}{n - p - t - \frac{1}{2}} * \frac{n - p - 2t}{4} \end{aligned} \tag{2.3.1}$$

We now wish to obtain bounds on $g(n, t, p)$ that only depend on n and t .

PROPOSITION 2.3.5. *For all $n \geq 0$, $t \geq 0$, and $p \geq 0$*

$$\frac{n - 2t}{4} < g(n, t, p) \leq \frac{n - 2t + 6}{4} \leq \frac{n + 6}{4}$$

We obtain a lower bound as follows:

$$\begin{aligned}
g(n, t, p) &= \frac{t+p+1}{p+t-\frac{1}{2}} * \frac{p}{4} + \frac{n-p-t+1}{n-p-t-\frac{1}{2}} * \frac{n-p-2t}{4} \\
&> \frac{t+p+1}{p+t+1} * \frac{p}{4} + \frac{n-p-t+1}{n-p-t+1} * \frac{n-p-2t}{4} \\
&= \frac{p}{4} + \frac{n-p-2t}{4} \\
&= \frac{n-2t}{4}
\end{aligned} \tag{2.3.2}$$

To obtain an upper bound on $g(n, t, p)$ we need the following claims:

Claim 1: For $t \geq 0$ and $p \geq 0$,

$$\frac{t+p+1}{p+t-\frac{1}{2}} * \frac{p}{4} \leq \frac{p+1}{p-\frac{1}{2}} * \frac{p}{4}$$

Proof. Note this is clearly true for $p = 0$. For $p > 1$ consider the inequality

$$\frac{t+p+1}{p+t-\frac{1}{2}} * \frac{p}{4} \leq \frac{p+1}{p-\frac{1}{2}} * \frac{p}{4}$$

Divide both sides by $\frac{p}{4}$ and cross multiply to obtain the following equivalent statement.

$$p^2 - \frac{t}{2} + pt + \frac{p}{2} - \frac{1}{2} \leq p^2 + t + pt + \frac{p}{2} - \frac{1}{2} \iff -\frac{t}{2} \leq t,$$

which is true.

Claim 2: For $p \geq 0$,

$$\frac{p+1}{p-\frac{1}{2}} * \frac{p}{4} \leq \frac{p+3}{4}$$

Proof. For $p \geq 1$,

$$\begin{aligned}
\frac{p+1}{p-\frac{1}{2}} * \frac{p}{4} &\leq \frac{p+3}{4} \iff \\
p^2 + p &\leq p^2 + \frac{5p}{2} - \frac{3}{2} \iff \\
0 &\leq \frac{3p}{2} - \frac{3}{2},
\end{aligned}$$

which is true for $p \geq 1$.

For $p = 0$, the inequality from Claim 2 is trivially true.

Thus, from Claims 1 and 2, we have shown that for the first term in $g(n, t, p)$,

$$\begin{aligned} & \frac{p+t+1}{2} * \frac{p}{2p+2t-1} \\ &= \frac{p+t+1}{p+t-\frac{1}{2}} * \frac{p}{4} \\ &\leq \frac{p+3}{4} \end{aligned}$$

Similarly, for the second term in $g(n, t, p)$, we have:

$$\begin{aligned} & \frac{o+t+1}{2} * \frac{o}{2o+2t-1} \\ &= \frac{o+t+1}{o+t-\frac{1}{2}} * \frac{o}{4} \\ &\leq \frac{o+3}{4} \end{aligned}$$

Therefore for the quantity $g(n, t, p)$, we have

$$\begin{aligned} g(n, t, p) &= \frac{p+t+1}{2} * \frac{p}{2p+2t-1} + \frac{o+t+1}{2} * \frac{o}{2o+2t-1} \\ &\leq \frac{p+3}{4} + \frac{o+3}{4} \\ &= \frac{p+o+6}{4} \\ &= \frac{n-2t+6}{4} \end{aligned} \tag{2.3.3}$$

We have shown the following:

$$\frac{n-2t}{4} \leq g(n, t, p) \leq \frac{n-2t+6}{4}$$

2.4. Bounding the Expected Number of Loop Edges from Below

Recall that $f(n, t, p) = \binom{n}{2t} \binom{n-2t}{p} \binom{2t+2p}{t+p} \binom{2n-2t-2p}{n-t-p} \frac{1}{(t+p+1)} \frac{1}{(n-t-p+1)}$. Also recall that the lower bound on the expected number of loop edges is $\frac{n-2t}{4}$. This leads us to the

following theorem:

THEOREM 2.4.1. *The lower bound of the expected number of loop edges $L(n)$ in a rooted 4-regular Hamiltonian planar graph of n vertices has asymptotic order*

$$L(n) \sim \frac{n}{6}$$

In order to prove the theorem, we recall from Section (1.5) that

$$\begin{aligned} i) \quad X(n) &= \sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p) \\ ii) \quad X(n) &\sim \frac{4}{\pi} \frac{12^n}{n^3} \end{aligned} \tag{2.4.1}$$

$$iii) \quad f(n, t, p) \geq \frac{12\sqrt{3}}{\pi^2} \frac{12^n}{n^4} * \frac{1}{e^{(9x^2+3y^2)/n}} \tag{2.4.2}$$

PROOF. We begin with the expected number of loop edges in all rooted Hamiltonian 4-regular planar graphs of n vertices:

$$L(n) = \frac{\sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p) g(n, t, p)}{\sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p)}$$

Replacing the denominator by its asymptotic order from Theorem (1.5.1), we obtain

$$L(n) \sim \frac{\pi n^3}{4 * 12^n} \sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p) g(n, t, p)$$

Now we replace $g(n, t, p)$ by its lower bound from Equation (2.3.2) to obtain

$$L(n) = \frac{\pi n^3}{4 * 12^n} \sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p) \frac{n-2t}{4}$$

We know that $f(n, t, p)$ has an absolute maximum for $t = \frac{n}{6}$ and $p = \frac{n}{3}$ [8]. To obtain a lower bound we will restrict our summation to certain terms around the maximum, throwing away the rest. The terms that we will consider are from $\frac{n}{6} - n^{\frac{3}{5}}$ to

$\frac{n}{6} + n^{\frac{3}{5}}$ for the t indices and from $\frac{n}{3} - n^{\frac{3}{5}}$ to $\frac{n}{3} + n^{\frac{3}{5}}$ for the p indices. Notice that in the factor that replaced $g(n, t, p)$, we now replace t by its maximally displaced value, $\frac{n}{3} + n^{\frac{3}{5}}$. Because of this, and since we are not considering all of the terms in the summation, the inequality holds that:

$$L(n) \geq \frac{\pi n^3}{4 * 12^n} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} f(n, \frac{n}{6} + x, \frac{n}{3} + y) * \frac{(n - 2(\frac{n}{6} + n^{3/5}))}{4}$$

We now replace $f(n, \frac{n}{6} + x, \frac{n}{3} + y)$ by its lower bound from Equation (2.4.2). This leads us to the following:

$$\begin{aligned} L(n) &\sim \frac{\pi n^3}{16 * 12^n} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} \frac{12\sqrt{3}}{\pi^2} \frac{12^n}{n^4} * \frac{1}{e^{(9x^2+3y^2)/n}} * \left(\frac{2n}{3} - 2n^{3/5} \right) \\ &= \frac{3\sqrt{3}}{4\pi} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} e^{-(9x^2+3y^2)/n} * \left(\frac{2}{3} - 2n^{-2/5} \right) \end{aligned}$$

Since $n^{-2/5} \rightarrow 0$, as n grows very large, the $n^{-2/5}$ term does not affect the asymptotic behavior of $L(n)$; hence, it can be dropped in the following steps, giving us

$$\begin{aligned} L(n) &\sim \frac{3\sqrt{3}}{4\pi} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} \frac{2}{3} e^{-(9x^2+3y^2)/n} \\ &= \frac{\sqrt{3}}{2\pi} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} e^{-(9x^2+3y^2)/n} \\ &= \frac{\sqrt{3}}{2\pi} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} e^{-9x^2/n} e^{-3y^2/n} \end{aligned}$$

The first factor in the double summation does not depend on y , so we factor it out of the inner sum. Then the two summations are independent, so they can be separated, and we now have

$$L(n) \sim \frac{\sqrt{3}}{2\pi} \left(\sum_{|x| \leq n^{\frac{3}{5}}} e^{-9x^2/n} \right) \left(\sum_{|y| \leq n^{\frac{3}{5}}} e^{-3y^2/n} \right) \quad (2.4.3)$$

Now consider only the first summation. Because $e^{-9x^2/n}$ is a positive and decreasing sequence for $x > 0$, we have

$$\int_1^{n^{3/5}} e^{-9x^2/n} dx < \sum_{x=1}^{n^{3/5}} e^{-9x^2/n} < \int_0^{n^{3/5}} e^{-9x^2/n} dx$$

Similarly,

$$\int_{-n^{3/5}}^{-1} e^{-9x^2/n} dx < \sum_{x=-n^{3/5}}^{-1} e^{-9x^2/n} < \int_{-n^{3/5}}^0 e^{-9x^2/n} dx$$

We now combine the middle and third parts of these two inequalities. In doing so, we pick up a 1 from the case when $x = 0$ in the sum. We also note the difference between the first and third parts of the inequalities. Thus we have

$$\left| \sum_{|x| \leq n^{\frac{3}{5}}} e^{-9x^2/n} - \int_{-n^{3/5}}^{n^{3/5}} e^{-9x^2/n} dx \right| \leq 1 + \int_{-1}^1 e^{-9x^2/n} dx \leq 3$$

It is also clear that

$$\int_{-n^{3/5}}^{n^{3/5}} e^{-9x^2/n} dx \leq \int_{-\infty}^{\infty} e^{-9x^2/n} dx$$

Thus, we have

$$\sum_{|x| \leq n^{\frac{3}{5}}} e^{-9x^2/n} \leq C + \int_{-\infty}^{\infty} e^{-9x^2/n} dx \text{ for some constant } C.$$

Thus

$$\sum_{|x| \leq n^{\frac{3}{5}}} e^{-9x^2/n} \sim \int_{-\infty}^{\infty} e^{-9x^2/n} dx \quad (2.4.4)$$

From Equations (2.4.3) and (2.4.4), we have

$$L(n) \sim \frac{\sqrt{3}}{2\pi} \int_{-\infty}^{\infty} e^{-9x^2/n} dx \int_{-\infty}^{\infty} e^{-3y^2/n} dy$$

To evaluate these integrals, we let $u = \sqrt{\frac{9}{n}}x$, $du = \sqrt{\frac{9}{n}}dx$, $v = \sqrt{\frac{3}{n}}y$ and $dv = \sqrt{\frac{3}{n}}dy$.

Then

$$\begin{aligned} L(n) &= \frac{\sqrt{3}}{2\pi} * \frac{\sqrt{n}}{3} \int_{-\infty}^{\infty} e^{-u^2} du \frac{\sqrt{n}}{\sqrt{3}} \int_{-\infty}^{\infty} e^{-v^2} dv \\ &= \frac{\sqrt{3}}{2\pi} \frac{n}{3\sqrt{3}} \int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-v^2} dv \end{aligned}$$

The two integrals are both Gaussian integrals, each of whose values can be evaluated exactly as $\sqrt{\pi}$ [12], giving

$$L(n) \sim \frac{\sqrt{3}}{2\pi} * \frac{n\pi}{3\sqrt{3}} = \frac{n}{6}$$

Thus we have shown, the lower bound of the expected number of loop edges $L(n)$ has asymptotic order

$$L(n) \sim \frac{n}{6}$$

□

2.5. Bounding the Expected Number of Loop Edges from Above

In this section, we will determine the upper asymptotic order of the expected number of loop edges in a rooted 4-regular Hamiltonian planar graph. The following theorem will be shown:

THEOREM 2.5.1. *Given n , the upper bound of the expected number of loop edges $L(n)$ in a rooted Hamiltonian 4-regular planar graph of n vertices has asymptotic order*

$$L(n) \sim \frac{n}{6}$$

PROOF. As before, we begin with the expected number of loop edges $L(n)$, replacing the denominator with its asymptotic order, and replacing $g(n, t, p)$ by its upper bound from Equation (2.3.3). This gives us the following

$$L(n) \sim \frac{\pi n^3}{4 * 12^n} \sum_{t=0}^{\frac{n}{2}} \sum_{p=0}^{n-2t} f(n, t, p) \left(\frac{n - 2t + 6}{4} \right)$$

We now reindex the double summation as in Section (2.4); however, we now consider all values for x and y . We also split the double summation into two sums. Replacing t by its minimum value, $\frac{n}{6} - n^{3/5}$, gives us the inequality:

$$\begin{aligned} L(n) \leq & \underbrace{\frac{\pi n^3}{4 * 12^n} \sum_{|x| \leq n^{\frac{3}{5}}} \sum_{|y| \leq n^{\frac{3}{5}}} f(n, \frac{n}{6} + x, \frac{n}{3} + y) * \frac{(n - 2(\frac{n}{6} - n^{3/5}) + 6)}{4}}_A \\ & + \underbrace{\frac{\pi n^3}{4 * 12^n} \sum_{c \geq n^{\frac{3}{5}}} f(n, \frac{n}{6} + x, \frac{n}{3} + y) * \frac{(n - 2(\frac{n}{6} - n^{3/5}) + 6)}{4}}_B \end{aligned}$$

Sum A can be handled as in Section (2.4). The difference between sum A and the sum in Section (2.4) is that term which now replaces $g(n, t, p)$ will produce $+2n^{-2/5} + 6n^{-1}$. Both of these terms go to zero as n grows very large, so they are not considered in the asymptotic value of the sum. As before, we find that

$$A \sim \frac{n}{6}$$

Sum B may be estimated as follows:

$$\begin{aligned}
B &= \frac{\pi n^3}{4 * 12^n} \sum_{|x| \text{ or } |y| \geq n^{\frac{3}{5}}} f(n, \frac{n}{6} + x, \frac{n}{3} + y) * \frac{(n - 2(\frac{n}{6} - n^{3/5}) + 6)}{4} \\
&= \frac{\pi n^3}{16 * 12^n} \sum_{|x| \text{ or } |y| \geq n^{\frac{3}{5}}} f(n, \frac{n}{6} + x, \frac{n}{3} + y) * (\frac{2n}{3} + 2n^{3/5} + 6)
\end{aligned}$$

Within the summation, $f(n, \frac{n}{6} + x, \frac{n}{3} + y)$ is maximized when either $|x|$, $|y|$, or both $|x|$ and $|y|$ are equal to $n^{3/5}$. Thus, we use the same approximation as before

$$L(n) \sim \frac{\pi n^3}{16 * 12^n} \sum_{|x| \text{ or } |y| \geq n^{\frac{3}{5}}} \frac{12\sqrt{3}}{\pi^2} \frac{12^n}{n^4} * \frac{1}{e^{(9x^2+3y^2)/n}} * (\frac{2n}{3} + 2n^{3/5} + 6)$$

In this sum, there are fewer than n^2 terms. So we now replace the sum by n^2 times the maximal term of the sum to obtain

$$\begin{aligned}
L(n) &\leq \frac{\pi n^3}{16 * 12^n} n^2 \frac{12\sqrt{3}}{\pi^2} \frac{12^n}{n^4} \frac{1}{e^{3n^{1/5}}} (\frac{2n}{3} + 2n^{3/5} + 6) \\
&= \frac{3\sqrt{3}}{4\pi} n e^{-3n^{1/5}} (\frac{2n}{3} + 2n^{3/5} + 6)
\end{aligned}$$

Asymptotically, this expression has the same order as

$$\frac{3\sqrt{3}}{4\pi} n^2 e^{-3n^{1/5}} \sim 0$$

As n grows very large, the above term goes to zero, due to the exponential factor.

Thus we have shown,

$$L(n) \sim \frac{n}{6} + 0 = \frac{n}{6}$$

□

From Sections 2.4 and 2.5, we have shown that the expected number of loop edges $L(n)$ in a rooted 4-regular Hamiltonian planar graph has asymptotic order

$$L(n) \sim \frac{n}{6}$$

CHAPTER 3

Quality of Bounds and Open Questions

3.1. Quality of Bounds

The result of Chapter 2 has been confirmed by numerical evidence. The algorithm described in Chapter 1 was used to generate samples of rooted 4-regular Hamiltonian planar graphs and the number of loop edges were counted. Table 3.1.1 shows the average of the number of loop edges in a random sample of several thousand randomly generated graphs for various values of n [14]. It is clear from the table that the average number of loop edges is very close to $\frac{n}{6}$.

n	$\overline{L(n)}$	$n / 6$
64	11.1104	10.66
128	21.8478	21.33
256	42.7038	42.66
512	85.635	85.33
1024	171.2034	170.66
2048	342.458	341.33
4096	686.334	682.66
8192	1365.56	1365.33
16384	2725.51	2730.66
32768	5474.55	5461.33

TABLE 3.1.1. A table displaying the average number of loop edges in a random sample of several thousand randomly generated graphs for various values of n , which is very close to $\frac{n}{6}$.

3.2. Open Questions

There are several open questions related to this research. A *4-edge-connected graph* G is a graph that can not be disconnected by removing any three edges in G [13]. We would like to get a count on the number of 4-edge-connected standard diagrams for a fixed number of vertices n . However, we can see that removing the edges incident to a loop edge from the Hamiltonian cycle H in G , will disconnect G . This can be seen in Figure 13. So any graph containing a loop edge is then 2-edge-connected.

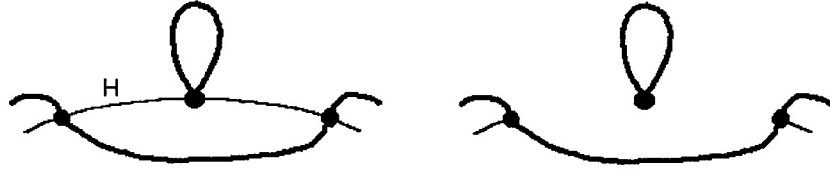


FIGURE 13. In this figure, if we delete the two edges on H incident to the vertex producing the loop edge, we obtain the figure on the right, which is disconnected.

Figure 14 shows an example of a *ring component*. A ring component consists of two edges u and v , one inside and one outside of H , where u and v are not on H and u and v have two common transition vertices as endpoints. It is of interest how the number of ring components grows with n since a ring component only contributes to the knot complexity in a very predictable way. Similar to the loop edges, one is interested in the knot or link that is obtained by deleting all ring components. In particular, we may ask if the number of ring components is $O(n)$, similar to the loop edges. It may be possible to use the same methods to estimate the number of ring

components whose transition vertices are adjacent on H as those used to estimate the number of loop edges.

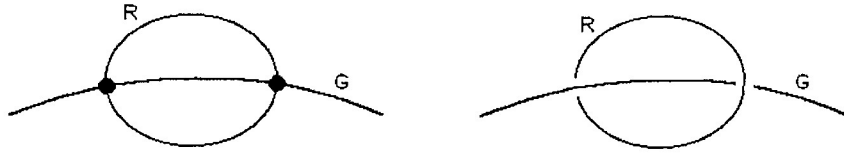


FIGURE 14. The ring labeled R will form a separate component when this portion of the diagram is viewed as an alternating knot, as shown on the right.

Figure 15 shows a ring component where the TVs are not adjacent on H . When the ring R is viewed as an alternating knot diagram, we can see that it can be moved to form a situation that is similar to that in Figure 14. Counting the number of ring components where the TVs are not adjacent will require different methods than the ones used to count the expected number of loop edges.

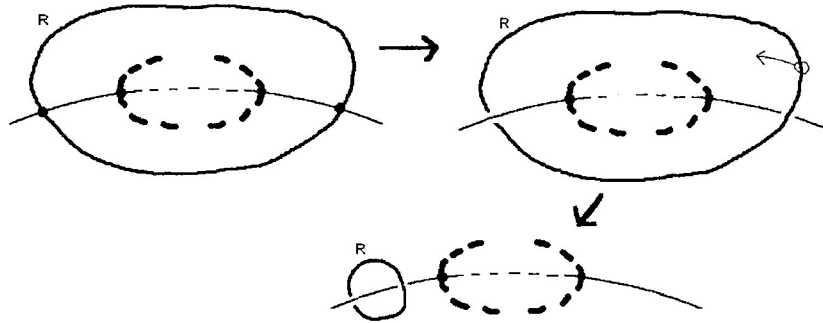


FIGURE 15. The original ring R can be contracted to a simpler situation when it is viewed as an alternating knot diagram, as shown in the figure.

If we remove the two Hamiltonian edges leading into and away from a ring component, the diagram becomes disconnected. Thus a 4-edge-connected graph G can not have any ring components.

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